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Quantized self-dual Maxwell field on a null surface

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The canonical formalism for a self-dual Maxwell field on a null plane is reviewed. After solution of the second class constraints, the transition to the quantum theory is carried out using a representation in which the self-dual Maxwell field is diagonal. The Gauss law constraint allows us to consider the physical state vectors to be holomorphic functionals of one complex function. Application of reality conditions allows us to define an inner product such that the Hermitean adjoint operators are identified with the classical complex conjugate operators. In going over to the Fourier expansion of the operators, we find that the inner product is formally convergent for positive frequency functionals and formally divergent for the negative frequency functionals. Following similar results of Ashtekar, Rovelli, and Smolin, negative frequency states are functional distributions identified with the helicity opposite to that of the positive frequency states.

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1. Introduction

A major program is underway to construct a quantum theory of gravity using the new variables introduced by Ashtekar [1–4]. While studying how to make use of these new variables in the full non-linear theory, application has been made both to the electromagnetic field [5] and to linearized general relativity [6]. A quantum theory, however, requires a complete set of independent variables from which one can construct the observables of the theory. It is well known that the identification of such quantities is most easily carried out on a null surface. For example, in the work of Bondi [7] and Sachs [8], one specifies the conformal two-geometry on space-like cuts of the outgoing null cone and at null infinity one gives information about the outgoing radiation together with a specification of the mass aspect and dipole aspect at one time. The latter contain information about the sources of the gravitational field whereas the former contain information about the independent degrees of freedom of the gravitational field. A similar analysis holds in the Newman–Penrose formalism [9,10].

For this reason, there have been attempts in the past to develop a canonical formalism for general relativity on a null surface. Torre [11] based his analysis on tetrads adapted to the two-by-two formalism of d'Inverno and Stachel [12]. An alternative approach based on the work of Bondi and Sachs was used by the present author [13]. In both cases only the classical theory was investigated. The results were of such complexity that no further progress toward a quantum theory was undertaken.

Based on the ideas of the Ashtekar variables, a new attempt to study the canonical formalism for general relativity on a null cone has begun [14,15]. However, in order to gain some familiarity with the use of self-dual variables, a preliminary study of the electromagnetic field has also been undertaken [16,17]. The previous studies are only of the classical field theory. Here I wish to present some results with respect to quantization. As in ref. [16], the electromagnetic field is considered on a null plane in Minkowski space. The vector potential, U(1) connection, is considered to be a real field, but only the self-dual components of the electromagnetic field appear in the Lagrangian. Because we are using a null surface as the base manifold on which to define the canonical formalism, second class constraints occur. These second class constraints are eliminated prior to passage to the quantum theory. The transition to the quantum theory is carried out in a representation in which the self-dual Maxwell field is diagonal, yet the two modes of polarization occur explicitly. The program carried out here is related to that in the paper "Self duality and quantization" by Ashtekar, Rovelli, and Smolin (henceforth referred to as ARS), which appears in this volume.

In the following section, previous work [16] is reviewed and extended. Then in section 3, the quantum theory is formulated. Only the free field is considered. A brief discussion of conclusions is in section 4.

2. Classical formalism

Following ref. [16], we introduce the coordinates

$$(u, v, \zeta, \bar{\zeta}) = ((t-z), \tfrac{1}{2}(t+z), (x+iy)/\sqrt{2}, (x-iy)/\sqrt{2}) ,$$

which are adapted to the null plane $u=$constant. In these coordinates, the natural basis forms a null tetrad all of whose connection coefficients vanish and the Minkowski space metric has the components $\eta_{01} = -\eta_{23} = 1$. Thus, we need not distinguish between coordinate and tetrad indices. (Lower case Greek indices will have the range 0–3 while lower case Latin indices will have the range 1–3.)

Duality of the Maxwell field is defined by $F^{*\rho\sigma} = \frac{1}{2}\epsilon^{\rho\sigma\iota\kappa}F_{\iota\kappa}$, $\epsilon^{0123} = \epsilon_{0123} = -\mathrm{i}$. Thus, we can introduce the self-dual two form $\mathscr{F} = \frac{1}{2}(F - \mathrm{i}F^*)$ so that $\mathscr{F}^* = \mathrm{i}\mathscr{F}$. Since there are only three independent self-dual components, we use the notation

$$\mathscr{F}^{01} = \mathscr{F}^{23} =: \mathscr{F}^1 , \qquad \mathscr{F}^{03} =: \mathscr{F}^3 , \qquad \mathscr{F}^{21} =: B . \tag{2.1}$$

Below, upper case Latin indices will take the values 1 and 3.

The first order action can be expressed in terms of the self-dual Maxwell field and the real connection (vector potential) as [3]

$$S=\frac{1}{2}\int [\mathrm{d}A\wedge\mathscr{F}-\tfrac{1}{4}\mathscr{F}\wedge\mathscr{F}] , \tag{2.2}$$

which takes the expanded form

$$S=\int \mathrm{d}u\int \mathrm{d}^3x\,\{\dot{A}_A\mathscr{F}^A+A_0\mathscr{F}^A{}_{,A}$$
$$+\mathscr{F}^1(2A_{[3,2]}-\mathscr{F}^1)+B(\mathscr{F}^3+2A_{[1,2]})\} . \tag{2.3}$$

In the above we have used $\mathrm{d}u\,\mathrm{d}^3x=\mathrm{i}\,\mathrm{d}u\wedge\mathrm{d}v\wedge\mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}$. The configuration space for this action is defined by the six variables $(A_A, \mathscr{F}^A, A_2, B)$. To go over to the Hamiltonian, we note that $\mathscr{F}^A$ is conjugate to A_A while the momenta conjugate to the remaining variables A_2 and B are zero, thus giving us primary constraints:

$$\Pi=0 , \qquad \pi=0 . \tag{2.4}$$

In addition, A_0 behaves like a Lagrangian multiplier whose variation gives an additional constraint.

The Hamiltonian now has the form

$$H=\int \mathrm{d}^3x\,\{\mathscr{F}^1(2A_{[2,3]}+\mathscr{F}^1)+B(2A_{[2,1]}-\mathscr{F}^3)-A_0\mathscr{F}^A{}_{,A}+\alpha\Pi+\beta\pi\} . \tag{2.5}$$

Propagation of the primary constraints and variation of the Lagrange multiplier A_0 lead to the secondary constraints

$$\{\Pi(x), H\}=\chi:=\mathscr{F}^1{}_{,3}+B_{,1}=0 , \tag{2.6a}$$

$$\{\pi(x), H\}=\phi:=-\mathscr{F}^3+A_{2,1}-A_{1,2}=0 , \tag{2.6b}$$

$$\mathscr{G}'(x):=\mathscr{F}^A{}_{,A}=0 . \tag{2.6c}$$

With the addition of a term in Π, (2.6c) has a vanishing Poisson bracket with the other four constraints and thus is first class,

$$\mathscr{G}:=\mathscr{F}^A{}_{,A}+\Pi_{,2} , \tag{2.7}$$

while the remaining four constraints are second class. Their Poisson brackets with each other do not vanish.

According to Dirac [18,19] second class constraints should be eliminated before quantization of the field. That is because they generate transformations which lead out of the space of solutions and should not become operators on the physical states. This elimination can be either through direct solution of the constraints themselves or by use of the Dirac bracket formalism [19,20]. In the present example, the direct solution is possible and therefore preferable. Equation (2.6a) can be solved for B and eq. (2.6b) for A_2:

$$B=B_0(\zeta,\bar{\zeta})-\int_{-\infty}^{v}\mathcal{F}^1{}_{,3}\,\mathrm{d}v' \,, \tag{2.8a}$$

$$A_2=A^0{}_2+\int_{-\infty}^{v}(A_{1,2}+\mathcal{F}^3)\,\mathrm{d}v' \,. \tag{2.8b}$$

These equations together with the primary constraints, eq. (2.4), eliminate four of the phase space variables. We are then left with A_A, $\mathcal{F}^A$, and the Gauss law first class constraint. Since each first class constraint permits the imposition of two restrictions on the phase space, there are two free complex functions remaining. The reality conditions will reduce these to two real functions.

While A_A and $\mathcal{F}^A$ continue to satisfy the standard Poisson bracket relations,

$$\{A_A(x),\mathcal{F}^B(x')\}=\delta_A{}^B(x-x') \,, \tag{2.9}$$

A_2 and B now have the following relations:

$$\{A_2(x),\mathcal{F}^1(x')\}=S(v-v')\delta_{,\zeta}(\zeta-\zeta')\delta(\bar{\zeta}-\bar{\zeta}') \,, \tag{2.10a}$$

$$\{A_2(x),A_3(x')\}=S(v-v')\delta(\zeta-\zeta')\delta(\bar{\zeta}-\bar{\zeta}') \,, \tag{2.10b}$$

$$\{B(x),A_1(x')\}=S(v-v')\delta(\zeta-\zeta')\delta_{,\bar{\zeta}}(\bar{\zeta}-\bar{\zeta}') \,. \tag{2.10c}$$

$S(v)$ is the unit step function, $S(v)=1$ for $v>0$, and $S(v)=0$ for $v<0$.

The reality of the connection implies that $A_3=\bar{A}_2$. Thus

$$A_3=\bar{A}^0{}_2+\int_{-\infty}^{v}(A_{1,3}+\bar{\mathcal{F}}^3)\,\mathrm{d}v' \,. \tag{2.11}$$

Equations (2.8)–(2.11) lead to the Poisson bracket relations

$$\{\bar{\mathcal{F}}^3(x),\mathcal{F}^3(x')\}=\delta_{,v}(v-v')\delta(\zeta-\zeta')\delta(\bar{\zeta}-\bar{\zeta}') \,, \tag{2.12a}$$

$$\{\bar{\mathcal{F}}^3(x),\mathcal{F}^1(x')\}=-\delta(v-v')\delta(\zeta-\zeta')\delta_{,\bar{\zeta}}(\bar{\zeta}-\bar{\zeta}') \,, \tag{2.12b}$$

$$\{\bar{\mathcal{F}}^1(x),\mathcal{F}^1(x')\}=S(v-v')\delta_{,\zeta}(\zeta-\zeta')\delta_{,\bar{\zeta}}(\bar{\zeta}-\bar{\zeta}') \,. \tag{2.12c}$$

The last of these relations follows from consistency with the Gauss law constraint. Note that, although we started from a complex Lagrangian which contained only components of the self-dual Maxwell field, use of the reality conditions introduces the anti-self-dual components.

By making use of the Gauss law constraint, the Hamiltonian can be written in the suggestive form

$$H=\int\mathrm{d}^3x\,\{\mathcal{F}^1\bar{\mathcal{F}}^1-A\mathcal{F}^A{}_{,A}\} \,. \tag{2.13}$$

Except for the contribution from the divergence term, the Hamiltonian is positive definite. In the source free case that we are considering, the divergence vanishes. Using the bracket expressions in eqs. (2.12), this Hamiltonian gives the correct equations of motion for A_1, $\mathscr{F}^1$, and $\mathscr{F}^3$. A, of course, fixes the value of A_0, which is completely arbitrary. The remaining variables are all determined from these.

3. Quantization

In the passage to the quantum theory we shall follow the program discussed in ref. [4]. This program is outlined in the ARS article included in this volume. The one difference in the application of the program to the present situation is that we have second class as well as first class constraints. According to Dirac [18,19], the second class constraints are not to be carried over to the quantum theory because they do not generate invariant transformations. That is, second class constraints do not map solutions onto solutions. Therefore, A_A and $\mathscr{F}^A$ are the basic variables whose algebra we wish to carry over to the quantum theory. These variables become the basic operators $\hat{A}_A$ and $\hat{\mathscr{F}}^A$. On the other hand, $\hat{A}_2$ (B will not be of interest) is defined in terms of the basic operators from eqs. (2.8b),

$$\hat{A}_2 = \int_{-\infty}^{v} \mathrm{d}v\,(\hat{A}_{1,2} + \hat{\mathscr{F}}^3)\,. \tag{3.1}$$

For the construction of the star algebra, the adjoints of these operators are defined by their complex conjugate relations on the classical phase space. In particular, we find that

$$\hat{\mathscr{F}}^{1\dagger} := \hat{\mathscr{F}}^1 - \hat{A}_{3,2} + \hat{A}_{2,3} \qquad \hat{\mathscr{F}}^{3\dagger} := \hat{A}_{3,1} - \hat{A}_{1,3}\,, \tag{3.2a,b}$$

$$\hat{A}_1^\dagger := \hat{A}_1\,, \qquad \hat{A}_3^\dagger := \hat{A}_2\,. \tag{3.2c,d}$$

The star algebra for the basic operators and their adjoints is equal to $i\hbar$ times the Poisson brackets for the basic variables and their conjugates given in the previous section. Thus,

$$[\hat{A}_A(x), \mathscr{F}^A(x')] = i\hbar\delta^3(x-x')\,, \tag{3.3a}$$

$$[\hat{\mathscr{F}}^{3\dagger}(x), \hat{\mathscr{F}}^3(x')] = i\hbar\delta_{,v}(v-v')\delta(\zeta-\zeta')\delta(\bar{\zeta}-\bar{\zeta}')\,, \tag{3.3b}$$

$$[\hat{\mathscr{F}}^{3\dagger}(x), \hat{\mathscr{F}}^1(x')] = -i\hbar\delta(v-v')\delta(\zeta-\zeta')\delta_{,\bar{\zeta}}(\bar{\zeta}-\bar{\zeta}')\,, \tag{3.3c}$$

$$[\hat{\mathscr{F}}^3(x), \hat{\mathscr{F}}^{1\dagger}(x')] = i\hbar\delta(v-v')\delta_{,\zeta}(\zeta-\zeta')\delta(\bar{\zeta}-\bar{\zeta}')\,, \tag{3.3d}$$

$$[\hat{\mathscr{F}}^{1\dagger}(x), \hat{\mathscr{F}}^1(x')] = i\hbar S(v-v')\delta_{,\zeta}(\zeta-\zeta')\delta_{,\bar{\zeta}}(\bar{\zeta}-\bar{\zeta}')\,. \tag{3.3e}$$

The adjoint operators are understood to be the expressions given in eqs. (3.2).

We shall assume that the operators $\hat{\mathscr{F}}^A$ are diagonal so that

$$\hat{\mathscr{F}}^A\Psi[\mathscr{F}^A]=\mathscr{F}^A\Psi[\mathscr{F}^A]\,,\qquad \hat{A}_A\Psi[\mathscr{F}^A]=\mathrm{i}\hbar\frac{\delta}{\delta\mathscr{F}^A}\Psi[\mathscr{F}^A]\,. \tag{3.4}$$

The physical states are those which are annihilated by the first class constraint, the Gauss law constraint:

$$\hat{\mathscr{F}}^A{}_{,A}\Psi[\mathscr{F}^A]=\mathscr{F}^A{}_{,A}\Psi[\mathscr{F}^A]=0\,. \tag{3.5}$$

This implies that the quantities in the argument of the physical state vector satisfy the Gauss law constraint. Therefore, the physical states may be considered to be functionals of $\mathscr{F}^3$ only and the action of $\hat{\mathscr{F}}^1$ to be derived from that of $\hat{\mathscr{F}}^3$. Thus,

$$\hat{\mathscr{F}}^3=F^3\Psi[\mathscr{F}^3] \tag{3.6a}$$

implies that

$$\hat{\mathscr{F}}^1\Psi[\mathscr{F}^3]=-\frac{\partial}{\partial\zeta}\Phi^3\Psi[\mathscr{F}^3]\,, \tag{3.6b}$$

where

$$\Phi^3(x)=\int_{-\infty}^{v}\mathrm{d}v'\ \mathscr{F}^3(v',\zeta,\bar{\zeta})\,. \tag{3.6c}$$

As a result, only gauge invariant operators can act on the physical states. Thus, $\hat{A}_1$ and $\hat{A}_3$ can act only in the gauge invariant form which defines $\hat{\mathscr{F}}^{3\dagger}$. From the commutation relations in (3.3b) it follows that

$$\hat{\mathscr{F}}^{3\dagger}\Psi[\mathscr{F}^3]=\mathrm{i}\hbar\frac{\partial}{\partial v}\frac{\delta}{\delta\mathscr{F}^3}\Psi[\mathscr{F}^3]\,. \tag{3.7}$$

The inner product on the space of physical states is determined by the requirement that the adjoint operators be, in fact, the Hermitean adjoint operators. Thus,

$$\overline{(\mathscr{X},\hat{\mathscr{F}}^3(x)\Psi)}=(\Psi,\hat{\mathscr{F}}^{3\dagger}\mathscr{X})\,. \tag{3.8}$$

More explicitly, this becomes

$$\int \mathrm{dI}\bar{\mathscr{F}}^3\wedge\mathrm{dI}\bar{\mathscr{F}}^3\ \mathscr{X}[\mathscr{F}^3]\mu[\mathscr{F}^3,\bar{\mathscr{F}}^3]\overline{\bar{\mathscr{F}}^3\Psi[\mathscr{F}^3]}$$
$$=\int \mathrm{dI}\mathscr{F}^3\wedge\mathrm{dI}\bar{\mathscr{F}}^3\ \mu[\mathscr{F}^3,\bar{\mathscr{F}}^3]\overline{\Psi[\mathscr{F}^3]}\hat{\mathscr{F}}^{3\dagger}\mathscr{X}[\mathscr{F}^3]\,. \tag{3.9}$$

Using eq. (3.6a) on the left hand side and (3.7) on the right, we find that the measure on the physical states, $\mu[\mathscr{F}^3,\bar{\mathscr{F}}^3]$, satisfies the following condition:

$$\mathrm{i}\hbar \frac{\partial}{\partial v} \frac{\delta}{\delta \mathscr{F}^3(x)} \mu[\mathscr{F}^3, \bar{\mathscr{F}}^3] = \bar{\mathscr{F}}^3 \mu[\mathscr{F}^3, \bar{\mathscr{F}}^3] \,. \tag{3.10}$$

The solution of this equation is

$$\mu[\mathscr{F}^3, \bar{\mathscr{F}}^3] = \exp\left(- \frac{\mathrm{i}}{2\hbar} \int \mathrm{d}^3x \, \{\bar{\Phi}^3(x) \mathscr{F}^3(x) - \bar{\mathscr{F}}^3(x) \Phi^3(x)\} \right), \tag{3.11}$$

where $\Phi^3(x)$ is defined in eq. (3.6c).

Since we are now free to make use of the reality conditions, the Hamiltonian now takes the form of eq. (2.13),

$$H = \int \mathrm{d}^3x \, \hat{\mathscr{F}}^1 \hat{\mathscr{F}}^{1\dagger} \,, \tag{3.12}$$

where the operators are understood to be expressed in terms of $\hat{\mathscr{F}}^3$ and $\mathscr{F}^{3\dagger}$. The action of $\hat{\mathscr{F}}^1$ is given in eq. (3.6) and

$$\hat{\mathscr{F}}^{1\dagger} = \hat{\mathscr{F}}^1 + 2\hat{A}_{[2,3]} = -\mathrm{i}\hbar \frac{\partial}{\partial \zeta} \frac{\delta}{\delta \mathscr{F}^3} \,. \tag{3.13}$$

To understand this Hamiltonian better, we shall carry out a Fourier expansion of the operators. Thus

$$\hat{\mathscr{F}}^3(x) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{-\infty}^{\infty} \mathrm{d}^3k \, \hat{f}^3(k) \mathrm{e}^{ik\cdot x} \,. \tag{3.14}$$

Equation (3.3b) then gives us the only commutator we shall need:

$$[\hat{f}^{3\dagger}(k), \hat{f}^3(k')] = -\hbar k_1 \delta^3(k-k') \,. \tag{3.15}$$

The transformed Hamiltonian has the form

$$H = \int \mathrm{d}^3k \, \frac{k_2 k_3}{k_1^2} \hat{f}^3(k) \hat{f}^{3\dagger}(k) \,. \tag{3.16}$$

Since $\mathscr{F}^3$ satisfies the wave equation in terms of the coordinates u, v, $\zeta\bar{\zeta}$, we have the condition on the frequency, $\omega k_1 + k_2 k_3 = 0$. Being the product of complex conjugates, $k_2 k_3$ is positive. Therefore, positive frequency is associated with negative k_1. Therefore, we define creation and annihilation operators as follows [21]:

$$a^\dagger(k) = \frac{1}{\sqrt{\hbar |k_1|}} \hat{f}^3(-k) \,, \quad a(k) = \frac{1}{\sqrt{\hbar |k_1|}} \hat{f}^{3\dagger}(-k) \,, \quad k_1 < 0 \,,$$

$$b^\dagger(k) = \frac{1}{\sqrt{\hbar |k_1|}} \hat{f}^{3\dagger}(k) \,, \quad b(k) = \frac{1}{\sqrt{\hbar |k_1|}} \hat{f}^3(k) \,, \quad k_1 > 0 \,. \tag{3.17}$$

Then, with normal ordering, the Hamiltonian takes the form

$$H=\int_0^\infty \mathrm{d}k_1 \int\int \mathrm{d}k_2\, \mathrm{d}k_3\, \hbar\omega[a^\dagger(k)a(k)+b^\dagger(k)b(k)]\,, \tag{3.18}$$

with $\omega=k_2k_3/k_1$. It is clear that $a^\dagger(k)$ and $a(k)$ create and annihilate right handed photons while $b^\dagger(k)$ and $b(k)$ create and annihilate left handed photons. Thus, although we started only with the self-dual field, the reality conditions and the use of both positive and negative frequencies allow us to describe both polarization states of the electromagnetic field.

However, the measure takes the form

$$\mu[f^3,\bar{f}^3]=\exp\left(\frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{\mathrm{d}^3k}{k_1}\bar{f}^3(k)f^3(k)\right), \tag{3.19}$$

which is convergent for positive frequencies and divergent for negative frequencies. In ARS, the same result occurs for fields defined on space-like surfaces. There they conclude that the negative frequencies are associated with state vectors which are distributions in the self-dual representation. Furthermore, they shows that in an anti-self-dual representation, the roles of the distributional and polynomial states would be reversed. This is in keeping with the interpretation above that the positive frequencies should be associated with one helicity and the negative frequencies with the other.

4. Conclusion

In the classical theory, the real source-free Maxwell field requires the specification of four real functions on a space-like Cauchy surface to determine the unique propagation of the field. That is, one gives four pieces of data per space point. This results, as is well known, in two states of polarization or helicity. On a null surface, on the other hand, only two pieces of data per hypersurface point – two functions – need be specified to describe the propagation of the field [22,23]. The difference results from the need to provide two additional functions on a surface at null infinity to have unique propagation. While we have been careful about boundary conditions in the classical description, in the present work we have not considered the quantization of the degrees of freedom at null infinity. This means that we are not considering the quantization of fields which may be parallel to the initial surface. Nonetheless, we obtain the propagation of two helicity states, which represents the two degrees of freedom of the electromagnetic field.

Had we used a future directed null cone rather than a null plane, there would have been no need for additional functions at null infinity for a discussion of the future propagation of the field. However, propagation into the past would require additional information. In this case, the additional information would represent the outgoing radiation. Therefore, a complete treatment of the quantized Maxwell field based on a null surface description needs to consider the quantization of fields at null infinity. Since with this paper we are trying to understand some of the difficulties to be faced in quantization of the gravitational field, this problem will have to be addressed.

In the case of the gravitational field, a representation in terms of the holonomy around arbitrary loops is being studied [24]. In fact, this representation has been worked out in considerable detail for the linearized theory [6] as well as for the Maxwell field on a space-like surface [5]. One can treat the holonomy on a null surface as well. This has been done for the example studied here. However, the resulting T-algebra is much more complicated because of causal connections along the null rays. The complications are such that a simple application of the loop representation does not look promising on the null surface.

There are other possible ways of looking for a suitable representation. One is through the work of Kozameh and Newman [25]. Another, which has been around for some time, but has not been applied in this connection, is the use of twistor theory through the Penrose transform [26]. Since the transform defines self-dual fields, it may be easier to apply on a null surface than the loop representation.

This work has benefited from my discussions with Chrys Soteriou and David Robinson about using a self-dual connection to describe canonical general relativity on a null surface. For the quantum discussion I am indebted to Abhay Ashtekar, who let me see the ARS manuscript before it was submitted and who helped to clarify some of the problems I found. Finally, it is a pleasure to submit this paper in honor of Roger Penrose. He has taught us so much about the behavior of massless fields that this work owes him a great deal.

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